

The Sutured Thurston Norm

JOHN CANTWELL
LAWRENCE CONLON

Department of Mathematics
St. Louis University
St. Louis, MO 63103
and
Department of Mathematics
Washington University
St. Louis, MO 63130

Email: CANTWELLJC@SLU.EDU and LC@MATH.WUSTL.EDU

Abstract

For sutured three-manifolds M , there is a *sutured* Thurston norm x^s due to M. Scharlemann [10]. Here, we show how depth one foliations of M can be useful tools for computing this norm. This uses the relation of these foliations with fibrations of DM (the double of M along the manifold $R \subset \partial M$ given by the sutured structure). We also prove and use the fact that a natural doubling map $D_* : H_2(M, \partial M) \rightarrow H_2(DM, \partial DM)$ is “norm doubling” with respect to the norms x^s and x on $H_2(M, \partial M)$ and $H_2(DM, \partial DM)$, respectively. All of this implies significant relations between the foliation cones of [5] and the sutured norm but, in general, these relations are difficult to pin down.

AMS Classification numbers Primary: 57R30

Secondary: 57M25, 57N10

Keywords: fibration, foliation, depth one, Thurston norm, sutured manifold, double

1 Introduction

If M is a compact 3-manifold, Thurston [11] defines a (semi)norm x on the real vector space $H_2(M, \partial M)$ (coefficients \mathbb{R} will be understood throughout), with unit ball polyhedral, and proves:

Theorem 1.1 *The fibrations of M over the circle that are transverse to ∂M correspond up to isotopy to the rays through lattice points in the open cones over certain top dimensional faces (called fibered faces) of the unit ball of the Thurston norm.*

The cones over fibered faces of the Thurston ball will be called *fibration cones*. This is slightly misleading since the classes lying in the interior of fibration cones correspond to foliations without holonomy, “most” of which are dense-leaved.

Let (M, γ) be a compact, connected, oriented, sutured 3-manifold [8]. Write

$$\partial M = \partial_\tau M \cup \partial_{\mathfrak{h}} M.$$

This notation, introduced in earlier papers of ours and in [1], anticipates a foliation tangent to $\partial_\tau M$ and transverse to $\partial_{\mathfrak{h}} M$. Wherever these parts of ∂M meet, M has a convex corner. This notation relates to the standard sutured manifold notation as follows:

$$\begin{aligned}\partial_{\mathfrak{h}} M &= \gamma = A(\gamma) \cup T(\gamma), \\ \partial_\tau M &= R(\gamma) = R_+ \cup R_-.\end{aligned}$$

Here, $A(\gamma)$ is a union of annuli and $T(\gamma)$ is a union of tori, while R_\pm are, respectively, the outwardly and inwardly oriented portions of $R(\gamma)$. The choice of orientations is part of the sutured structure and each component of R_- is separated from a component of R_+ by annular components of γ . Finally, each suture is a closed curve in the interior of a component of $A(\gamma)$, parallel to and oriented with the boundary curves of this annulus. The union of the sutures is denoted by s .

We will be interested in taut foliations of M , hence will require that M be irreducible and, as a sutured manifold, *taut*. This latter requirement means that each component of $\partial_\tau M$ is norm-minimizing in $H_2(M, \partial_{\mathfrak{h}} M)$. In particular, if $\sigma \subset \partial_\tau M$ is an imbedded loop bounding a disk in M , it also bounds a disk in $\partial_\tau M$.

In [5], we proved the following analog of Thurston’s Theorem (Theorem 1.1) for depth one foliations.

Theorem 1.2 *Let (M, γ) be a compact, connected, oriented, irreducible, taut, sutured 3-manifold. There are finitely many closed, convex, polyhedral cones in $H_2(M, \partial M)$, called foliation cones, having disjoint interiors and such that the taut, transversely oriented, depth one foliations of (M, γ) that are transverse to $\partial_{\cap} M$ and have the components of $\partial_{\tau} M$ as sole compact leaves correspond to the rays through integer lattice points of $H_2(M, \partial M)$ in the interior of the foliation cones.*

Remark Set $M_0 = M \setminus \partial_{\tau} M$ and remark that a depth one foliation as above restricts to a fibration of M_0 over the circle. The classes in the interior of foliation cones that are not on rays through integer lattice points correspond to foliations “almost without holonomy” with each leaf in M_0 dense in M .

Remark It is known [3] that the “foliated ray” $\langle \mathcal{F} \rangle$ corresponding to the depth one foliation \mathcal{F} determines \mathcal{F} up to a C^0 isotopy that is smooth in M_0 .

Remark In contrast to Thurston’s result, the cones in Theorem 1.2 are generally not defined by a norm. Indeed, they are not generally symmetric with respect to multiplication by -1 .

Remark The proof of Theorem 1.2 in [5] had some serious gaps. The authors are preparing a revised version [6] of that paper that resolves these problems.

There is a seminorm x^s for sutured manifolds, called the *sutured* Thurston norm. This is due to M. Scharlemann [10] and, if $s = \emptyset$, x^s reduces to the usual norm x . In this note we develop ideas relating x^s to the depth one foliations classified by Theorem 1.2 and show how this theory can be used to compute the norm. This makes the computations of the norm, done in the examples at the end of [5], rigorous. In those examples, the foliation cones are unions of cones over some faces of the Thurston ball of x^s , but this fails in Example 2 of the present paper. However, even in this example, x^s is closely enough related to the foliation cones that we are able to compute the Thurston norm.

2 Doubling

There are three basic topics to be treated here, namely: the doubling map in singular homology, the Thurston norm in sutured manifolds and their doubles, and the process of inducing fibrations in the double DM from certain depth one foliations on M .

2.1 The Doubling Map

If M is a smooth, connected, oriented, sutured manifold, we form the double DM along $\partial_\tau M$ (assumed to be nonempty). This is defined in complete analogy with the usual definition of the double of a manifold along its full boundary. Thus DM is an oriented manifold formed by taking a second copy of M , but with opposite orientation, and gluing the two together via the identity map on $\partial_\tau M$. We write

$$DM = M \cup (-M)/\sim.$$

There is a standard way to put a smooth, oriented structure on DM so that ∂DM is also smooth and so that the natural reflection map $\rho : DM \rightarrow DM$ is an orientation-reversing diffeomorphism. This map interchanges the corresponding points of M and $-M$, hence has $\partial_\tau M$ as its set of fixed points.

Let $S \subset M$ be a smooth, properly imbedded, oriented surface. Reversing orientations gives $-S \subset -M$. The double $DS = S \cup (-S) \subset DM$ can be viewed as an oriented, properly imbedded submanifold of DM . There is a technical problem that, if $S \cap \partial_\tau M \neq \emptyset$, smoothness of DS might fail along this set. To avoid this, one introduces a ρ -invariant Riemannian metric on DM . There is a ρ -invariant normal neighborhood U of $\partial_\tau M$ in DM and an isotopy of S makes $S \cap U$ saturated by the normal fibers of $U \cap M$. Now DS is a smooth, ρ -invariant subsurface of DM . Of course, if $S \cap \partial_\tau M = \emptyset$, DS is the disjoint union of S and $-S$. Note also that $\rho|_{DS}$ is an orientation-reversing diffeomorphism of this surface.

A smooth triangulation of S determines a smooth triangulation of DS , producing singular cycles mod the boundary in M and DM respectively. The corresponding classes $[S] \in H_2(M, \partial M)$ and $[DS] \in H_2(DM, \partial DM)$ are well defined, independently of the choice of triangulation. We will define a canonical “doubling” map

$$D_* : H_2(M, \partial M) \rightarrow H_2(DM, \partial DM)$$

such that $D_*[S] = [DS]$ and show that this map is “norm doubling”.

At the level of singular chains, the map $\rho|M : M \rightarrow DM$ induces a linear map

$$\rho_\# : C_\#(M, \partial M) \rightarrow C_\#(DM, \partial_\tau M \cup \partial DM)$$

commuting with the singular boundary operator $\partial_\#$. Thus, we can define

$$D_\#(c) = c - \rho_\#(c), \quad \forall c \in C_\#(M, \partial M),$$

noting that this also commutes with $\partial_\#$. At this point, there is a small technical problem. The map D_* induced by $D_\#$ takes its image in the space

$H_*(DM, \partial_\tau M \cup \partial DM)$, whereas we want to interpret it as a map into the space $H_*(DM, \partial DM)$. The crucial property to notice is that, if the singular chain c is supported in $\partial_\tau M$, then $D_\#(c) = 0$.

Consider the open cover $\Phi = \{U, V\}$ of DM , where $U = \text{int } DM$ and V is a normal neighborhood of ∂DM with normal fibers along $\partial(\partial_\tau M)$ lying entirely within $\partial_\tau M$. Let $A = \partial_\tau M \cap V$ and note that ∂DM is a deformation retract of $A \cup \partial DM$. By abuse of notation, we also let Φ denote the induced open cover on any subspace of DM and we compute singular homology on DM and any of its subspaces using the Φ -small singular chain complex $C_\#^\Phi$. That is, each singular simplex in a chain $c \in C_\#^\Phi$ is supported either in U or in V . It is standard that the Φ -small homology H_*^Φ is canonically equal to the ordinary singular homology H_* , the equality being induced by $C_\#^\Phi \subset C_\#$.

If $c \in C_\#^\Phi(\partial M)$, then, since $D_\#$ annihilates all singular simplices in $\partial_\tau M$, $D_\#(c)$ is a chain on $A \cup \partial DM$. We obtain homomorphisms

$$\begin{aligned} D_\# : C_\#^\Phi(M) &\rightarrow C_\#^\Phi(DM) \\ D_\# : C_\#^\Phi(\partial M) &\rightarrow C_\#^\Phi(A \cup \partial DM), \end{aligned}$$

of chain complexes, hence a chain homomorphism

$$D_\# : C_\#^\Phi(M, \partial M) \rightarrow C_\#^\Phi(DM, A \cup \partial DM).$$

This defines

$$D_* : H_*(M, \partial M) \rightarrow H_*(DM, A \cup \partial DM) = H_*(DM, \partial DM),$$

the desired doubling map.

Remark The above supposes that $\partial_\tau M$ meets $\partial_\cap M$. Otherwise, $\partial_\cap M = T(\gamma)$ and the proof that $D_* : H_*(M, \partial M) \rightarrow H_*(DM, \partial DM)$ is even easier, not requiring the use of Φ -small homology.

Lemma 2.1 *If $S \subset M$ is a properly imbedded surface, then $D_*[S] = [DS]$.*

Proof Indeed, if $c_S \in Z_2(M, \partial M)$ is a fundamental cycle for S obtained by a smooth triangulation, it is an elementary consequence of the orientation-reversing property of $\rho : DS \rightarrow DS$ that $c_S - \rho_\#(c_S) \in Z_2(DM, \partial DM)$ is a fundamental cycle for DS . \square

Consider the inclusion map $i : M \hookrightarrow DM$ and the induced homomorphism

$$i^* : H^1(DM) \rightarrow H^1(M)$$

in real cohomology. Using Lefschetz duality, we view this as

$$i^* : H_2(DM, \partial DM) \rightarrow H_2(M, \partial M).$$

Lemma 2.2 *The composition $i^* \circ D_*$ is equal to the identity on $H_2(M, \partial M)$. In particular, the doubling map is injective on $H_2(M, \partial M)$.*

Proof It will be enough to prove this for elements $[S] \in H_2(M, \partial M)$, where S is a properly imbedded, oriented surface in M . Indeed, these constitute the integer lattice in $H_2(M, \partial M)$. By Lemma 2.1, we must show that $i^*[DS] = [S]$. The Lefschetz dual of $[DS]$ is represented by a 1-form ω as follows. Fix a normal neighborhood V of DS in DM . This can be chosen so that $V \cap \partial_\tau M$ is saturated by normal fibers, as is $V \cap \partial DM$. The closed form ω is supported in V and has integral along each normal fiber equal to 1. Evidently, $V \cap M$ is a normal neighborhood of S and ω restricts in M to a representative of the Lefschetz dual of $[S]$. \square

Remark It is easy to give a geometric definition of

$$i^* : H_2(DM, \partial DM) \rightarrow H_2(M, \partial M)$$

on each element $[\Sigma]$ of the integer lattice. Represent this class by a properly imbedded surface $\Sigma \subset DM$ that is transverse to $\partial_\tau M$ and note that $\Sigma_+ = \Sigma \cap M$ is a properly imbedded surface in M . Then $i^*[\Sigma] = [\Sigma_+]$.

2.2 The Thurston Norm

Roughly speaking, we define the Thurston norm in a sutured manifold by doubling along $\partial_\hbar M$, computing the Thurston norm in the doubled manifold, and dividing by two. This is half the norm defined by Scharlemann in [10, Definition 7.4].

More precisely, let S be properly imbedded as usual and connected. By a small isotopy, ∂S can be assumed to be transverse to $\partial \partial_\hbar M$ and we compute $\chi_-^s(S)$ by doubling along $\partial_\hbar M$, computing the usual χ_- of the doubled surface and dividing by two. (The superscript s stands for “sutured”.) One can give an intrinsic formula for this number as follows.

The components of $S \cap \partial_\hbar M$ are circles and/or properly imbedded arcs in annular components of $\partial_\hbar M$. These circles need not be essential and some of the arcs might also fail to be essential in the sense that they start and end on the same boundary component of an annular component in $\partial_\hbar M$. We will see that these inessential arcs and circles can be eliminated, but for the moment

they are allowed. Let $n(S)$ denote the number of arc components of $S \cap \partial_{\mathfrak{H}} M$. Then the reader can verify that the formula for χ_-^s is

$$\chi_-^s(S) = \begin{cases} -\chi(S) + \frac{1}{2}n(S), & \text{if this number is positive,} \\ 0, & \text{otherwise.} \end{cases}$$

As usual, if S is not connected, one defines $\chi_-^s(S)$ as the sum of the values on each component. If z is an element of the integer lattice in $H_2(M, \partial M)$, $x^s(z)$ is defined to be the minimum value of $\chi_-^s(S)$ taken over all surfaces $S \in z$. Continuing to follow Thurston's lead, we extend x^s canonically to a pseudonorm on the vector space $H_2(M, \partial M)$ and call this the *sutured* Thurston norm.

Remarks Instead of computing the sutured norm by doubling in $\partial_{\mathfrak{H}} M$, one can equally well double in $\partial_{\tau} M$. Again the components of $S \cap \partial_{\tau} M$ are properly imbedded arcs and/or circles and the number of arc components is the same number $n(S)$. One then notes that $2\chi_-^s(S) = \chi_-(DS)$, where $\chi_-(DS)$ is defined as for the ordinary Thurston norm.

We further remark that, by a χ_-^s -reducing homology and/or isotopy, S can be assumed to meet each annular component of $\partial_{\mathfrak{H}} M$ only in essential arcs, each crossing the suture once, or in essential circles, each parallel to the suture and disjoint from it. It can be assumed also that S meets each toral component only in essential circles, although this remark is not particularly consequential. At any rate, $n(S)$ is now just the number of times that ∂S crosses the sutures and it is elementary that this number is even. Thus, $\chi_-^s(S)$ is an integer, as is $x^s[S]$.

Example A decomposing disk Δ in the sense of Gabai [8] has $\chi_-^s(\Delta) = 0$ if it meets the sutures twice, $\chi_-^s(\Delta) = 1$ if it meets them four times, etc.

Theorem 2.3 *The map*

$$D_* : H_2(M, \partial M) \rightarrow H_2(DM, \partial DM)$$

is norm-doubling, where the sutured Thurston norm is used on the first space and the usual Thurston norm is used on the second. Thus, if B is the Thurston ball of M and B^ that of DM , then $D_*(B/2) = B^* \cap D_*(H_2(M, \partial M))$.*

Proof It is enough to prove this on elements of the integer lattice. Let $[S]$ be represented by a χ_-^s -minimal surface S . We have already noted that $\chi_-(DS) = 2\chi_-^s(S)$, hence it will be enough to show that DS is a χ_- -minimal representative of $[DS] = D_*[S]$. If not, let $\Sigma \in [DS]$ have $\chi_-(\Sigma) < \chi_-(DS)$.

Isotope Σ smoothly to be transverse to $\partial_\tau M$ and let $\Sigma_+ = \Sigma \cap M$ and $\Sigma_- = \Sigma \cap (-M)$. If no component of Σ_\pm has positive Euler characteristic, one verifies the relation

$$\chi_-(\Sigma) = \chi_-^s(\Sigma_+) + \chi_-^s(\Sigma_-). \quad (*)$$

The only possible components with positive Euler characteristic are spheres or disks. In the first case, irreducibility of M permits elimination of the offensive component. In the second, there will be no problem if the boundary of the disk Δ meets $\partial_\tau M$ in arcs. Otherwise, $\partial\Delta$ is a simple closed loop either in $\partial_{\text{th}}M$ or $\partial_\tau M$. In the first case, Δ is also a component of Σ in DM and has zero Thurston norm. In M it has zero sutured norm, so this case also causes no problem. In the remaining case, $\partial\Delta \subset \partial_\tau M$ and tautness of the sutured manifold structure, together with irreducibility, yields an isotopy of Σ pulling the disk Δ through $\partial_\tau M$, hence eliminating it as a component of Σ_\pm . Thus $(*)$ can be assumed to hold. Interchanging the roles of M and $-M$, if necessary, we can then assume that $\chi_-^s(\Sigma_+) < \chi_-^s(S)$. But

$$[S] = i^*[DS] = i^*[\Sigma] = [\Sigma_+],$$

contradicting χ_-^s -minimality of S in $[S]$. \square

2.3 Inducing Fibrations on DM

In this subsection, we assume that M , as a sutured manifold, is not a product $\partial_\tau M \times I$. This insures that $\partial_\tau M$ cannot be a fiber in a fibration of DM over the circle. We sketch some facts that are treated in greater detail in [3], [4] and [5].

Let \mathcal{F} be a smooth, depth one foliation of M , transverse to $\partial_{\text{th}}M$ and having the components of $\partial_\tau M$ as sole compact leaves. A depth one leaf $L \subset M_0$ determines an element $\lambda(\mathcal{F}) \in H^1(M; \mathbb{Z})$ of the integer lattice in the real cohomology space $H^1(M)$ via the intersection product with loops in M_0 . This class can also be represented by a closed, nonsingular 1-form ω on M_0 that “blows up nicely” at $\partial_\tau M$ (meaning that ω becomes unbounded near $\partial_\tau M$ in such a way that the 2-plane field $\ker \omega$ extends smoothly to a 2-plane field on M tangent to $\partial_\tau M$). The form ω defines $\mathcal{F}|_{M_0}$, hence also determines \mathcal{F} , and its cohomology class can be viewed as a class on M via the homotopy equivalence $M_0 \hookrightarrow M$ (the natural inclusion map). For any positive constant a , the form $a\omega$ also defines \mathcal{F} , so we obtain a “foliated ray” $\langle \mathcal{F} \rangle \subset H^1(M)$ corresponding to \mathcal{F} . This ray, in turn, determines \mathcal{F} up to an isotopy that is smooth in M_0 and continuous on M [3, Theorem 1.1]. We often think of a foliated ray as

an isotopy class of foliations. These foliated rays are exactly the rays meeting integer lattice points in the interiors of the foliation cones of [5].

Remark Poincaré duality identifies $H^1(M) = H_2(M, \partial M)$.

The leaves of $\mathcal{F}|M_0$ spiral in a well-understood way on each component F of $\partial_\tau M$, giving rise to a nondivisible cohomology class

$$\nu : \pi_1(F) \rightarrow \mathbb{Z}$$

called the *juncture* of the spiral (cf. [3, §3]). The juncture on F depends only on the class $\lambda(\mathcal{F})$ [3, Lemma 3.1]. It can be represented by a compact, properly imbedded, oriented, nonseparating 1-manifold $N \subset F$ which need not be connected [3, pp. 159–160] and each component is assigned an integer weight.

If there is a depth one foliation \mathcal{G} such that $\lambda(\mathcal{G}) = -\lambda(\mathcal{F})$, we will denote \mathcal{G} by $-\mathcal{F}$ and call this the *opposite* foliation to \mathcal{F} . Remark that this is not the foliation defined by the form $-\omega$, even up to isotopy, since this foliation would require that the outwardly oriented components of $\partial_\tau M$ become inwardly oriented and vice versa. These orientations are part of the given sutured structure on M and may not be reversed. While, in many cases, $-\mathcal{F}$ exists, examples show that it may not. Indeed, the three vertices in Figure 2 of Section 5 are not foliated classes, but they are the negatives of foliated classes. Of course, at the cohomology level, $[-\omega] = \lambda(-\mathcal{F})$. By the ideas in the proof of [3, Lemma 3.1], the juncture for $-\mathcal{F}$ can be represented by $-N$, the manifold obtained by reversing the orientation of N . Intuitively, the foliations \mathcal{F} and $-\mathcal{F}$ spin in “opposite directions” along F , appearing to be “mirror images” of one another in a small normal neighborhood of F in M .

Suppose that \mathcal{F} admits an opposite foliation $-\mathcal{F}$. We can produce a taut foliation $\mathcal{F} \cup -\mathcal{F}$ on DM by using \mathcal{F} in M and $-\mathcal{F}$ in $-M$, the components of $\partial_\tau M$ being the sole compact leaves. Since the foliation is taut, each of the compact leaves is a properly imbedded, incompressible surface in DM .

If F is one of these compact leaves, it inherits an orientation so that it is inwardly oriented with respect to M or $-M$ and outwardly oriented with respect to the other. Thus the junctures in F for the respective foliations can be taken to be physically the same submanifold of F , but with opposite orientations. It follows that the procedure in [4, pp. 379–381] applies, allowing us to erase these compact leaves by deleting their “spiral ramp” neighborhoods and fitting the resulting foliations together, matching convex corners of one to concave corners of the other and vice versa (cf. [4, Fig. 4]). Actually, our situation is a bit

more complicated than that envisioned in [4] because our juncture need not be connected, but essentially the same construction goes through. In this way we erase all leaves that are components of $\partial_\tau M$. The resulting foliation of DM , denoted by $D\mathcal{F}$, has only compact leaves since the construction amputates the finitely many ends of all leaves and joins together their compact cores. Thus, $D\mathcal{F}$ is a fibration of DM over the circle, the fibers being transverse to ∂DM . The reader should be warned that $D\mathcal{F}$ is not uniquely determined by \mathcal{F} and $-\mathcal{F}$. The topology of the fiber depends on the choices of spiral ramp neighborhoods of the components F of $\partial_\tau M$. With a little care, this construction can be carried out so that the following is true.

Lemma 2.4 *If the depth one foliation \mathcal{F} admits an opposite foliation, then there are associated fibrations $D\mathcal{F}$ of DM over the circle with fibers transverse to ∂DM . Furthermore, there is a smooth, one-dimensional foliation \mathcal{L} of DM , tangent to ∂DM and transverse both to $\mathcal{F} \cup -\mathcal{F}$ and $D\mathcal{F}$.*

While each component F of $\partial_\tau M$ fails to be a leaf of $D\mathcal{F}$, it remains an incompressible surface in DM with a special relationship to $D\mathcal{F}$.

Lemma 2.5 *The surface F is isotopic through properly imbedded surfaces in DM to a surface that has only positive saddle tangencies with $D\mathcal{F}$.*

Proof The tangent bundles $\tau = \tau(\mathcal{F} \cup -\mathcal{F})$ and $\tau_0 = \tau(D\mathcal{F})$ are both transverse to \mathcal{L} and transversely oriented so that both induce the same orientation along \mathcal{L} . It follows that τ and τ_0 are homotopic as oriented 2-plane bundles, hence have the same (relative) Euler class $e(\tau) = e(\tau_0) \in H^2(M, \partial M)$. Thus

$$\int_F e(\tau_0) = \int_F e(\tau) = \chi(F).$$

We can assume, via a small isotopy near ∂DM , that each component of ∂F is either transverse to $D\mathcal{F}$ or lies in a fiber of $D\mathcal{F}$. The two possibilities correspond, respectively, to the cases in which the component of ∂F does or does not meet the juncture for \mathcal{F} . Thus, Thurston's general position result [11, Theorem 4] allows us to perform an isotopy of F , putting it in a position so that all tangencies with $D\mathcal{F}$ are saddles. (The possibility that F could be isotoped onto a fiber is eliminated by our assumption that M is not a product.) If some tangency is not positive (that is, the orientations of $\tau(F)$ and $\tau(D\mathcal{F})$ at the tangency are opposite), it would follow that $\int_F e(\tau_0) \neq \chi(F)$, a contradiction. \square

Remark Lemma 2.5 can also be proven more directly by a Morse theoretic argument.

Proposition 2.6 *If the depth one foliation \mathcal{F} admits an opposite foliation and if $K \subset DM$ is a properly imbedded surface having only positive saddle tangencies with $D\mathcal{F}$, then $[K] \in H_2(DM, \partial DM)$ lies in the cone over a fibered face of the Thurston ball and K is a norm minimizing representative of $[K]$.*

Proof Let $C \subset H_2(DM, \partial DM)$ be the cone over a top dimensional face of the Thurston ball, the interior of which contains the “fibered ray” $\langle D\mathcal{F} \rangle$ associated to $D\mathcal{F}$ as in Theorem 1.1. Let $[D\mathcal{F}] \in \langle D\mathcal{F} \rangle \setminus \{0\}$. Then, by a standard argument of Thurston [11], the fact that the tangencies are positive saddles implies that the convex combination $t[D\mathcal{F}] + (1-t)[K] \in \text{int } C$, $0 < t \leq 1$. Consequently, $[K] \in C$. The norm x is linear in C , coinciding there with the linear functional $-e(\tau(D\mathcal{F})) : H_2(DM, \partial DM) \rightarrow \mathbb{R}$, and so

$$x([K]) = -e(\tau(D\mathcal{F}))([K]) = -\chi(K).$$

This latter equality is due to the fact that the tangencies are positive saddles [11] (see also [2, Lemma 10.1.13]). \square

Corollary 2.7 *If the depth one foliation \mathcal{F} admits an opposite foliation and if $F \subset DM$ is as in Lemma 2.5, then $[F]$ lies in the cone over a lower dimensional face of a fibered face of the Thurston ball and F is norm minimizing in $[F]$.*

Proof Indeed, by Proposition 2.6 and Lemma 2.5, F is norm minimizing in $[F]$ and that class lies in the cone over a fibered face. It cannot be in the interior of that cone since F is not the fiber of a fibration of DM . \square

Corollary 2.8 *If the depth one foliation \mathcal{F} admits an opposite foliation and if $S \subset M$ is a properly imbedded surface such that DS is smooth and has only positive saddle tangencies with $D\mathcal{F}$, then $x^s[S] = -\frac{1}{2}\chi(DS)$ and $S \in [S]$ realizes this minimal sutured norm.*

Proof Indeed, by Proposition 2.6, DS is norm minimizing in $[DS]$. The assertion follows by Lemma 2.1 and Theorem 2.3. \square

3 Sutured Handlebodies

Lemma 3.1 *There is a canonical decomposition*

$$H_2(DM, \partial DM) = H_2(M, \partial M) \oplus \ker i^*,$$

where $H_2(M, \partial M)$ is imbedded as the image of D_* .

Proof Since $i^* \circ D_*$ is the identity on $H_2(M, \partial M)$, this is immediate. \square

Lemma 3.2 $\ker i^* \cong H_2(M, \partial_{\text{th}} M)$.

Proof By the long exact cohomology sequence of the pair (DM, M)

$$H^0(DM) \xrightarrow{i^*} H^0(M) \xrightarrow{\partial^*} H^1(DM, M) \rightarrow H^1(DM) \xrightarrow{i^*} H^1(M) \cdots$$

and the fact that $i^* : H^0(DM) \rightarrow H^0(M)$ is an isomorphism, it follows that $\partial^*(H^0(M)) = 0$. Thus, the kernel of $i^* : H^1(DM) \rightarrow H^1(M)$ is isomorphic to $H^1(DM, M)$. By excision and homotopy invariance, this space is isomorphic to $H^1(-M, \partial_{\text{th}}(-M))$. There is no harm in dropping the minus sign and employing Lefschetz duality to identify this space with $H_2(M, \partial_{\text{th}} M)$. Here, the version of Lefschetz duality we are using is the seldom quoted one proven in [9, Theorem 3.43]. \square

Let M be a sutured handlebody of genus n . We will let γ_i , $1 \leq i \leq m$, denote the sutures and also the homology class each suture represents in $H_1(M)$. Let $\{\gamma'_i\}_{i=1}^m$ denote the basis of $H_1(\partial_{\text{th}} M)$ represented by these sutures. Let $X \subset M$ be a bouquet of circles $\alpha_j \subset M$, $1 \leq j \leq n$, that is a deformation retract of M . Viewing α_j as representing a homology class in $H_1(M)$ as well as a curve, one obtains a basis $\{\alpha_j\}_{j=1}^n$ of $H_1(M)$.

Consider the map

$$W : H_1(\partial_{\text{th}} M) \rightarrow H_1(M)$$

induced by the inclusion $\partial_{\text{th}} M \hookrightarrow M$.

Lemma 3.3 *The vector space $H_2(M, \partial_{\text{th}} M)$ is canonically imbedded in the vector space $H_1(\partial_{\text{th}} M)$ as $\ker W$.*

Proof This follows from the long exact sequence

$$\cdots \rightarrow 0 = H_2(M) \rightarrow H_2(M, \partial_{\text{th}} M) \xrightarrow{\partial} H_1(\partial_{\text{th}} M) \xrightarrow{W} H_1(M) \cdots$$

\square

Remark In the above long exact sequence, the map W can be represented by the $n \times m$ matrix

$$\mathbf{W} = \begin{bmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nm} \end{bmatrix}.$$

Here, we coordinatize $H_1(\partial_{\hbar}M)$ by the basis $\{\gamma'_i\}_{i=1}^m$ and $H_1(M)$ by $\{\alpha_j\}_{j=1}^n$. The columns of \mathbf{W} are the vectors γ_i , $1 \leq i \leq m$. The column rank r of this matrix is the rank of the linear map W and the dimension of the kernel of W is $d = m - r$.

Theorem 3.4 $H_2(DM, \partial DM) \cong H_2(M, \partial M) \oplus \mathbb{R}^d$.

Proof Indeed,

$$\begin{aligned} H_2(DM, \partial DM) &\cong H_2(M, \partial M) \oplus \ker i^* && \text{(Lemma 3.1)} \\ &\cong H_2(M, \partial M) \oplus H_2(M, \partial_{\hbar}M) && \text{(Lemma 3.2)} \\ &\cong H_2(M, \partial M) \oplus \ker W && \text{(Lemma 3.3)} \end{aligned}$$

□

Let c be the number of components of $\partial_{\tau}M = R_+ \cup R_-$.

Theorem 3.5 *One has $d \geq c - 1$, with equality if and only if the linear map W has rank $m - c + 1$ if and only if the identification in Lemma 3.1 is*

$$H_2(DM, \partial DM) = H_2(M, \partial M) \oplus \mathbb{R}^{c-1}.$$

If $d = c - 1$, the factor \mathbb{R}^{c-1} is generated by the classes represented by any $c - 1$ of the components of $R_+ \cup R_-$.

Proof The first equivalence follows since the rank of W equals $m - d$ while the second equivalence is immediate by Theorem 3.4. By Lemma 3.1, the factor \mathbb{R}^{c-1} is identified in $H_2(DM, \partial DM)$ as $\ker i^*$ and it is clear that each component N_i of $R_+ \cup R_-$ determines a homology class $\nu_i = [N_i] \in \ker i^*$. Thus, it will be sufficient to show that any $c - 1$ of these classes are linearly independent. This will also show that $d \geq c - 1$.

First note that the classes determined by the components of R_+ are linearly independent, as are those determined by the components of R_- . Indeed, there is a loop in DM having intersection number 1 with any given component of R_+ and intersection number 0 with all others. The same argument works for the

components of R_- , proving that there is no nontrivial linear relation between the classes corresponding to the components of one of R_\pm .

Next, choosing the indexing appropriately, let $\{\nu_i = [N_i]\}_{i=1}^{c-1}$ be a choice of $c - 1$ of the classes and let $\nu_c = [N_c]$ be the omitted one. For definiteness, suppose that N_c is a component of R_+ . We consider a linear relation

$$0 = \sum_{i=1}^{c-1} a_i \nu_i$$

and show that each a_i is forced to be zero. For each component N_i of R_- , there is an arc in M from N_c to N_i and this doubles to a loop in DM that has intersection number a_i with the right hand side of the above relation. Thus, $a_i = 0$ whenever N_i is a component of R_- . The above relation, therefore, involves only terms corresponding to components of R_+ . As already observed, there is no such nontrivial relation. An entirely similar argument works when N_c is a component of R_- . \square

Corollary 3.6 *The linear map W has rank $m - 1$ if and only if the identification in Lemma 3.1 is*

$$H_2(DM, \partial DM) = H_2(M, \partial M) \oplus \mathbb{R}.$$

In this case, the factor \mathbb{R} is generated by $[R_+] = [R_-]$ and both R_+ and R_- are connected.

Let g be the genus of $R_+ \cup R_-$.

Theorem 3.7 $m - c + 1 + g = n$.

Proof The disjoint union of R_+ and R_- has genus g . The proof consists of sequentially pasting together adjoining components of the disjoint union of R_+ and R_- along a common suture. This operation either reduces the number of components by one or adds a handle. The totality of such pastings produces a surface homeomorphic to ∂M , a connected surface of genus n . Since there are c components, $c - 1$ of the pastings along sutures reduce the number of components and the remaining $m - (c - 1)$ pastings add handles to give a total of $m - c + 1 + g$ handles. The assertion follows. \square

4 Computing the Sutured Thurston Norm

Our goal in this section is to state and prove a proposition that can often be used to find top dimensional faces of the Thurston ball. It applies to all the examples at the end of [5] and Example 2 of Section 5. We let $[\mathbf{a}_1, \dots, \mathbf{a}_n]$ denote the closed, convex hull of a set of points $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ in $H_2(M, \partial M)$ or $H_2(DM, \partial DM)$ and we let $\langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle$ be the cone with base $[\mathbf{a}_1, \dots, \mathbf{a}_n]$ and cone point $\mathbf{0}$.

Definition A *simple disk decomposition* of M is a complete disk decomposition of M in which all the disks are disjoint proper disks in M . That is we can assume all the disk are there at the beginning when we do the disk decomposition rather than having to do the disk decomposition sequentially.

The following lemmas are consequences of Gabai's procedure of disk decomposition [7]. If $D_i \subset M$ is a disk of a simple disk decomposition, we will denote the class $[D_i] \in H_2(M, \partial M)$ by \mathbf{e}_i .

Lemma 4.1 *If $\{+D_1, \dots, +D_n\}$ is a simple disk decomposition of M giving the depth one foliation \mathcal{F} , then each D_i , $1 \leq i \leq n$, meets \mathcal{F} in positive saddles. Furthermore, $\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$ is a subcone of a foliation cone and $\langle \mathcal{F} \rangle \setminus \{\mathbf{0}\} \subset \text{int } \langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$.*

For a proof, see [6, Corollary 2.8].

Lemma 4.2 *If $\{+D_1, \dots, +D_n\}$ is a simple disk decomposition of M giving the foliation \mathcal{F} , then $\{+D_1, \dots, +D_n\}$ is a simple disk decomposition of $-M$ giving the foliation \mathcal{F} . Each $D_i \subset -M$, $1 \leq i \leq n$, meets \mathcal{F} in positive saddles. Furthermore, $\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$ is a subcone of a foliation cone of $-M$.*

Proof Each D_i , $1 \leq i \leq n$, and \mathcal{F} have the opposite transverse orientation in $-M$ as in M , as does $R(\gamma)$. □

Lemma 4.3 *If $\{-D_1, \dots, -D_n\}$ is a simple disk decomposition of M giving the foliation \mathcal{F} , then each $-D_i$ meets \mathcal{F} in positive saddles and so the cone $\langle -\mathbf{e}_1, \dots, -\mathbf{e}_n \rangle = -\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$ is a subcone of a foliation cone of both M and $-M$.*

Proof Apply Lemma 4.1 and 4.2 □

In the following, a boundary component of a properly imbedded surface S is said to cross the sutures essentially if its intersections with annular components of $\partial_{\text{th}}M$ are essential arcs. Indeed, a small isotopy of S removes any inessential intersections of ∂S with sutures. When $S = D$ is a disk of a disk decomposition, the term “essentially” is redundant by Gabai’s definition of disk decomposition, but we will use it anyway for emphasis.

Proposition 4.4 *If $\{D_1, \dots, D_n\}$ and $\{-D_1, \dots, -D_n\}$ are simple disk decompositions of M , then there is a fibration $D\mathcal{F}$ of DM over the circle such that the surfaces $D_i \cup -D_i$, $1 \leq i \leq n$, and R_+ have only positive saddle tangencies with the fibration. Further the Thurston norm of $D_*\mathbf{e}_i = [D_i \cup -D_i] \in H_2(DM, \partial DM)$ is the number of times ∂D_i essentially crosses the sutures minus 2 and the sutured Thurston norm of \mathbf{e}_i is half this number.*

Proof The disk decomposition $\{D_1, \dots, D_n\}$ (respectively $\{-D_1, \dots, -D_n\}$) gives the subcone $\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$ of a foliation cone of M (respectively, it gives the subcone $\langle -\mathbf{e}_1, \dots, -\mathbf{e}_n \rangle$ of a foliation cone of $-M$). If $\langle \mathcal{F} \rangle \subset \text{int} \langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$, then $\langle -\mathcal{F} \rangle \subset \text{int} \langle -\mathbf{e}_1, \dots, -\mathbf{e}_n \rangle$. Then by Lemma 2.4, \mathcal{F} and $-\mathcal{F}$ can be matched up across $\partial_{\tau}M$ to give a fibration $D\mathcal{F}$. Further Lemmas 4.1 and 4.3 imply that $D_i \cup -D_i$, $1 \leq i \leq n$, has only positive saddle tangencies with $D\mathcal{F}$ while Lemma 2.5 implies that (after a small isotopy of DM moving $D\mathcal{F}$ and all $D_i \cup -D_i$) R_+ has only positive saddle tangencies with $D\mathcal{F}$.

Let b_i be the number of times ∂D_i essentially crosses the sutures. Then the surface $D_i \cup -D_i$ is a punctured sphere with b_i boundary components and thus $-\chi(D_i \cup -D_i) = b_i - 2$. Since this surface has only positive saddle tangencies with the fibration, Proposition 2.6 implies that $x(D_*\mathbf{e}_i) = b_i - 2$ and Corollary 2.8 implies that $x^s(\mathbf{e}_i) = x(D_*\mathbf{e}_i)/2$. \square

In the examples we are interested in, the matrix \mathbf{W} of Section 3 has rank $m - 1$ so, by Corollary 3.6, $\partial_{\tau}M$ has one positive component R_+ and one negative component R_- and $H_2(DM, \partial DM) = H_2(M, \partial M) \oplus \mathbb{R}$ where the \mathbb{R} factor is generated by $\mathbf{R} = [R_+] = [R_-]$, and, without loss, we can assume

$$D_*(H_2(M, \partial M)) = H_2(M, \partial M) \oplus \{0\} \subset H_2(DM, \partial DM).$$

In the following corollary, the integer m and the matrix \mathbf{W} are as in Section 3.

Corollary 4.5 *All four of the cones*

$$\langle D_*\mathbf{e}_1, \dots, D_*\mathbf{e}_n, \pm \mathbf{R} \rangle \text{ and } \langle D_*(-\mathbf{e}_1), \dots, D_*(-\mathbf{e}_n), \pm \mathbf{R} \rangle$$

are subcones of fibration cones (full-dimensional if $\text{rank } \mathbf{W} = m - 1$) and thus each lies in a cone over a fibered face of the Thurston ball of DM . Also, the sutured Thurston norm is linear on both the cones $\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle, \langle -\mathbf{e}_1, \dots, -\mathbf{e}_n \rangle \subset H_2(M, \partial M)$ and both of these cones are full-dimensional subcones of foliation cones and are contained in cones over top dimensional faces of the Thurston ball of M .

Proof Since $\{D_1, \dots, D_n\}$ and $\{-D_1, \dots, -D_n\}$ are simple disk decompositions of M and $-M$ respectively, Proposition 4.4 gives a fibration $D\mathcal{F}$ meeting the surfaces $D_i \cup -D_i$, $1 \leq i \leq n$, and R_+ in positive saddles. Thus, $\langle D_*\mathbf{e}_1, \dots, D_*\mathbf{e}_n, \mathbf{R} \rangle$ is a subcone of a fibration cone (Proposition 2.6). If $\text{rank } \mathbf{W} = m - 1$, this cone is full-dimensional by Corollary 3.6 and the fact that \mathbf{R} is not in the image of D_* . Similarly, since $\{-D_1, \dots, -D_n\}$ and $\{D_1, \dots, D_n\}$ are simple disk decompositions of M and $-M$ respectively, one sees that the cone $\langle D_*(-\mathbf{e}_1), \dots, D_*(-\mathbf{e}_n), \mathbf{R} \rangle$ is a subcone of a fibration cone (full-dimensional if $\text{rank } \mathbf{W} = m - 1$). One obtains the other two fibration cones because the Thurston ball and its fibered faces are symmetric under multiplication by -1 .

We prove the second part of the corollary for $\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$. The proof for the cone $-\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$ is identical. We must show that if $\mathbf{p} = u \cdot \mathbf{p}_1 + v \cdot \mathbf{p}_2$, with $\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2 \in \langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$ and $u, v \in \mathbb{R}$ then $x^s(\mathbf{p}) = u \cdot x^s(\mathbf{p}_1) + v \cdot x^s(\mathbf{p}_2)$. Suppose on the contrary that $x^s(\mathbf{p}) \neq u \cdot x^s(\mathbf{p}_1) + v \cdot x^s(\mathbf{p}_2)$. Then, by Theorem 2.3, $x(D_*\mathbf{p}) \neq u \cdot x(D_*\mathbf{p}_1) + v \cdot x(D_*\mathbf{p}_2)$. This contradicts the linearity of the Thurston norm over faces of the Thurston ball of DM .

Since the sutured Thurston norm is linear on $\langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$, this is an (obviously full-dimensional) subcone of the cone over a fibered face of the Thurston ball. It is also a subcone of a foliation cone by Lemma 4.1. \square

5 Examples

In many case we can figure out the Thurston ball of knot or link complements cut apart along the Seifert surface using the methods of Section 4. The methods of Example 1 can be used to make rigorous the computations of the Thurston norm in [5, §7].

Example 1 Let M be the complement of the pretzel link $(2, 2, 2)$ cut apart along its Seifert surface as in [5, §7, Example 1] (see Figure 1). One can do disk decompositions using disks $\{D_i, -D_j\}$ as long as $i \neq j \in \{0, 1, 2\}$. These

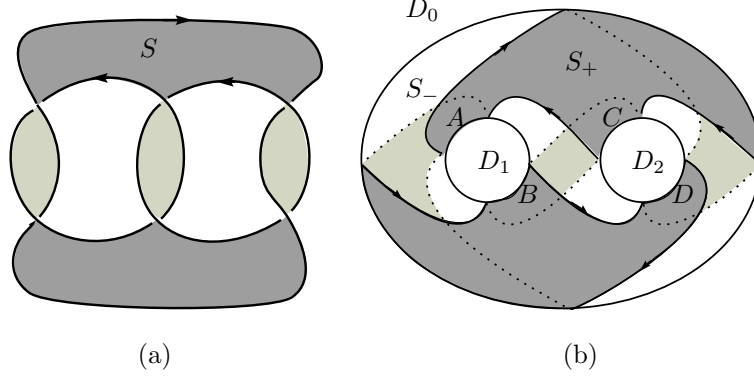


Figure 1: (a) A Seifert surface for $(2, 2, 2)$ (b) The sutured manifold M obtained from $(2, 2, 2)$

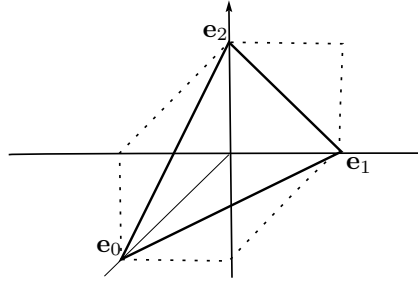


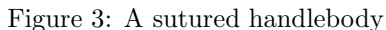
Figure 2: Thurston ball and foliation cones for $(2, 2, 2)$

disk decompositions are extremely easy to do using Gabai's graphical algorithm in [7, Theorem 6.1]. Since each of the ∂D_i 's essentially crosses the sutures 4 times, it follows from Proposition 4.4 that $x(D_* \mathbf{e}_i) = 2$ and $x^s(\mathbf{e}_i) = 1$. By Corollary 4.5, it follows that the Thurston ball is the dotted hexagon B of Figure 2.

The Markov process argument of [5, §7, Example 1 or Example 2] shows that $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$, $\langle \mathbf{e}_2, \mathbf{e}_0 \rangle$ and $\langle \mathbf{e}_0, \mathbf{e}_1 \rangle$ are the foliation cones.

Suitably labelling the sutures, we have that $\gamma_1 = -\alpha_1 + \alpha_2$, $\gamma_2 = \alpha_1 + \alpha_2$, and $\gamma_3 = \alpha_1 - \alpha_2$ in $H_1(M)$ (notation as in §3). The matrix

$$\mathbf{W} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$



Proposition 5.1 *The Thurston ball of DM is the double cone (suspension) over $D_*(B/2)$ with cone points $\pm \mathbf{R}$.*

Example 2 Regard Figure 3 as drawn on S^2 , the boundary of a solid ball \mathcal{B} . Paste D_1 to D_1 so that A (respectively B) on one copy of D_1 is matched to A (respectively B) on the other copy of D_1 and the sutures match up, paste D_2 to D_2 so that C (respectively D) on one copy of D_2 is matched to C (respectively D) on the other copy of D_2 and the sutures match up, and paste D_3 to D_3 so that E (respectively F) on one copy of D_3 is matched

to E (respectively F) on the other copy of D_3 and the sutures match up. Then Figure 3 represents a sutured handlebody M of genus 3 with sutures $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Clearly, $H_2(M, \partial M) = \mathbb{R}^3$.

The arrows on the disks in Figure 3 define the positive orientation of the disks. Let α be a simple closed curve in Figure 3 going once around D_1 , D_2 , and D_3 in the negative sense and essentially crossing the sutures γ_2 twice and γ_3 and γ_4 once each. Then α bounds an oriented disk in the solid ball \mathcal{B} which we will denote D_0 . In $H_2(M, \partial M)$, $\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$.

5.1 The Thurston Ball

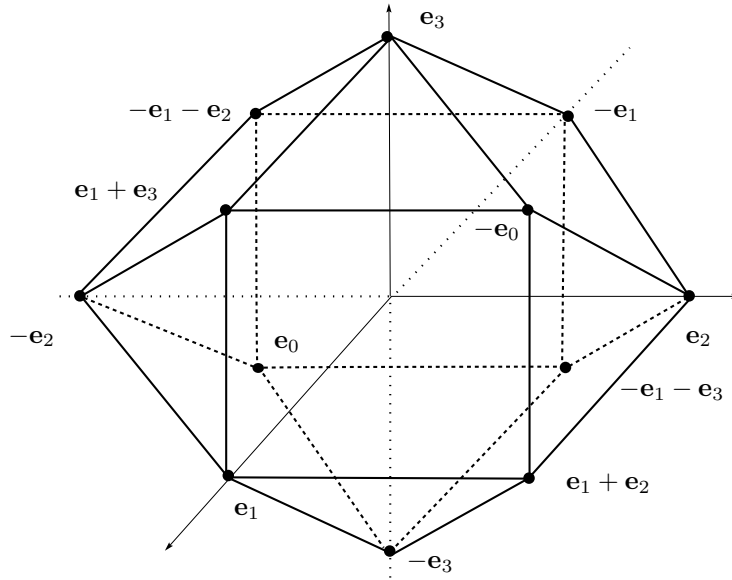


Figure 4: The Thurston ball B

Consider the compact, convex polyhedron depicted in Figure 4. One easily checks that the vertices of the quadrilateral faces really are coplanar. Two of these faces will present special problems in the following analysis.

Definition The two quadrilateral faces $Q^\pm = \pm[\mathbf{e}_2, \mathbf{e}_3, -\mathbf{e}_0, -\mathbf{e}_1]$ will be called the *exceptional faces*.

Lemma 5.2 *Each of the vertices in Figure 4 is represented by an oriented properly imbedded disk in M , the boundary of which essentially crosses the sutures four times.*

Proof This is clear for $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and has already been observed for \mathbf{e}_0 . For $\mathbf{e}_1 + \mathbf{e}_2$, draw a closed, positively oriented curve on $\partial\mathcal{B}$ meeting the suture γ_1 once, γ_2 twice, and γ_3 once. This bounds the desired disk in M . One argues similarly for $\mathbf{e}_1 + \mathbf{e}_3$, obtaining a disk with boundary meeting γ_1 once, γ_2 twice, and γ_4 once. The negatives of these classes are represented by the respective oppositely oriented disks. \square

Lemma 5.3 *The vertices in Figure 4 all have sutured Thurston norm one and the sutured norm is identically equal to 1 on each of the nonexceptional faces.*

Proof If $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ are any three vertices of a nonexceptional face with corresponding representative disks $\{\Delta_1, \Delta_2, \Delta_3\}$, then these disks and their negatives give simple disk decompositions and each of the disks has boundary that essentially crosses the sutures 4 times. Verifying these disk decompositions by Gabai's algorithm is routine but tedious. The lemma then follows by Proposition 4.4 and Corollary 4.5. \square

Let $\pm D_i$ denote the disk representing $\pm \mathbf{e}_i$, $0 \leq i \leq 3$. Then there are simple disk decompositions $\{-D_1, D_2, D_3\}$ and $\{-D_0, D_2, D_3\}$ and simple disk decompositions $\{D_0, D_1, -D_2\}$ and $\{D_0, D_1, -D_3\}$. There can be no pairs of simple disk decompositions $\{\Delta_1, \Delta_2, \Delta_3\}$ and $\{-\Delta_1, -\Delta_2, -\Delta_3\}$ that can be used in Corollary 4.5 to show that Q^\pm are faces. Instead we will show that $x^s(\mathbf{e}_2 + \mathbf{e}_3) = 2$, which proves, by convexity of the sutured Thurston ball, that Q^+ is a face. Of course, the norm of $-\mathbf{e}_2 - \mathbf{e}_3$ is also 2 and Q^- is a face.

In the following, $\partial(D_2 \cup D_3)$ and the sutures γ_i are viewed as 1-cycles on ∂M .

Lemma 5.4 *The intersection numbers of $\partial(D_2 \cup D_3)$ with the sutures is given by: $\gamma_1 \cdot \partial(D_2 \cup D_3) = -2$, $\gamma_2 \cdot \partial(D_2 \cup D_3) = 4$, $\gamma_3 \cdot \partial(D_2 \cup D_3) = -1$, $\gamma_4 \cdot \partial(D_2 \cup D_3) = -1$.*

Proof Let \mathbf{n} be an exterior normal to ∂M and use a right hand rule to define the intersection number $\gamma_i \cdot D_j$, i.e. $\gamma_i \cdot D_j = \pm 1$ depending on whether $(\gamma_i, D_j, \mathbf{n})$ is a right or left handed system $1 \leq i, j \leq 3$. One can compute the intersection numbers:

$$\begin{array}{llll} \gamma_1 \cdot \partial D_2 = -1 & \gamma_2 \cdot \partial D_2 = +2 & \gamma_3 \cdot \partial D_2 = -1 & \gamma_4 \cdot \partial D_2 = 0 \\ \gamma_1 \cdot \partial D_3 = -1 & \gamma_2 \cdot \partial D_3 = +2 & \gamma_3 \cdot \partial D_3 = 0 & \gamma_4 \cdot \partial D_3 = -1 \end{array}$$

The lemma follows. \square

Lemma 5.5 *If D is a properly embedded disk in M and ∂D crosses the sutures essentially at most twice, then D is boundary compressible. If S is a properly embedded, connected surface in M which is not a boundary compressible disk and whose boundary crosses the sutures essentially (and does so cross some sutures), then $\chi_-(DS) \geq 2$.*

Proof Suppose D is a properly embedded disk with ∂D meeting the sutures at most twice. Put D into general position with respect to D_1 , D_2 , and D_3 . The points of intersection of D with D_1 , D_2 , and D_3 will consist of circles and arcs. Assume the ends of the arcs do not lie on sutures.

By an innermost circle on D argument, we can get rid of all circles of intersection.

Similarly, by an innermost arc argument on D we can get rid of all arcs of intersection without increasing the number of intersections of ∂D with the sutures. In fact, choose an arc of intersection α in D having endpoints x and y such that there exists an arc $\beta \subset \partial D$ having endpoints x and y with $\alpha \cup \beta$ bounding a disk $D' \subset D$ such that $\text{int } D'$ meets none of the arcs in the innermost arc argument. Since there are at least two such α and β and since ∂D meets the sutures at most twice, we can assume α and β chosen so that β meets the sutures at most once. The arc α will be a properly embedded arc in D_{i_0} , some $1 \leq i_0 \leq 3$. Thus, there is an arc $\delta \subset \partial D_{i_0}$ with endpoints x and y , such that $\alpha \cup \delta$ bounds a disk $D'' \subset D_{i_0}$. Since ∂D_{i_0} meets the sutures four times and there are two possible choices of δ , we can assume δ meets the sutures at most twice. Thus $\delta \cup \beta$ is a simple closed curve in ∂M meeting the sutures at most three times, therefore never or twice. Therefore $\delta \cup \beta$ bounds a disk $D''' \subset \partial M$ (D''' lies on the sphere represented in Figure 3 and D''' contains none of $\pm D_j$, $1 \leq j \leq 3$) and a suture meets δ if and only if it meets β . Since M is irreducible, the sphere $D' \cup D'' \cup D'''$ bounds a ball that can be used to give an isotopy of D removing the arc of intersection α . Indeed, D' can be moved onto D'' , keeping α fixed, and then an arbitrarily small isotopy pulls this image of D' free of D_{i_0} . Since a suture meets δ if and only if it meets β , the isotopy does not change the number of intersections of ∂D with the sutures. After finitely many isotopies, we may assume that D does not meet D_i , $1 \leq i \leq 3$ and that ∂D meets the sutures at most twice. Cut M apart along D_1 , D_2 , and D_3 to give the solid ball \mathcal{B} with boundary S^2 (see Figure 3). Clearly, D is boundary compressible in the solid ball \mathcal{B} and so in M .

Thus if S has boundary meeting the sutures and S is not a boundary compressible disk with ∂S meeting the sutures twice, then either S is a disk with

∂S meeting the sutures 4 or more times, or S has genus $g \geq 1$, or S has at least 2 boundary components and S has genus $g = 0$. In the first case $\chi_-(DS) \geq 4 - 2 = 2$ and, in the second case, $\chi_-(DS) \geq 2 + 4g - 2 = 4g > 2$. The third case falls into two subcases. If only one boundary component meets $\partial_\tau M$, then DS has genus 0 and at least four boundary components, in which case $\chi_-(DS) \geq 4 + 0 - 2 = 2$. If at least two boundary components of S meet $\partial_\tau M$, then DS has genus at least 1 and at least two boundary components, hence $\chi_-(DS) \geq 2 + 2 - 2 = 2$. \square

Lemma 5.6 $x^s(\mathbf{e}_2 + \mathbf{e}_3) = 2$ and so $x^s \equiv 1$ on each of the exceptional faces Q^\pm .

Proof The double of $S = D_2 \cup D_3$ consists of two four times punctured spheres with Euler characteristic $2 \cdot (2 - 4) = -4$. Dividing by two we see that

$$x^s(\mathbf{e}_2 + \mathbf{e}_3) \leq \chi_-^s(S) \leq |-2| = 2.$$

Let S be a surface representing $[D_2 \cup D_3]$ in $H_2(M, \partial M)$. Thus $\chi_-^s(S) = \frac{1}{2}\chi_-(DS)$. By Lemma 5.4,

$$\gamma_1 \cdot \partial S = -2, \gamma_2 \cdot \partial S = 4, \gamma_3 \cdot \partial S = -1, \gamma_4 \cdot \partial S = -1.$$

Therefore, ∂S must meet the sutures at least eight times. If S has only one component S_1 whose boundary meets the sutures, then

$$\chi_-^s(S) \geq \chi_-^s(S_1) \geq \frac{1}{2}\chi_-(DS_1) \geq \frac{1}{2}(8 + 4g - 2) \geq 3,$$

where g is the genus of S_1 . Otherwise S has at least two components, S_1 and S_2 , whose boundaries meet the sutures. Thus, by Lemma 5.5,

$$\chi_-^s(S) \geq \frac{1}{2}\chi_-(DS_1) + \frac{1}{2}\chi_-(DS_2) \geq 2.$$

In any event, $x^s(\mathbf{e}_2 + \mathbf{e}_3) \geq 2$ and equality holds.

For the last assertion, the fact that $x^s = 1$ on $\pm(\mathbf{e}_2 + \mathbf{e}_3)/2$ and on each vertex of Q^\pm , together with convexity of the unit ball, implies that $x^s|_{Q^\pm} \equiv 1$. \square

Theorem 5.7 *The polyhedron B in Figure 4 is the unit ball of x^s .*

Indeed, by Lemma 5.3 and Lemma 5.6, $x^s \equiv 1$ on each of the faces.

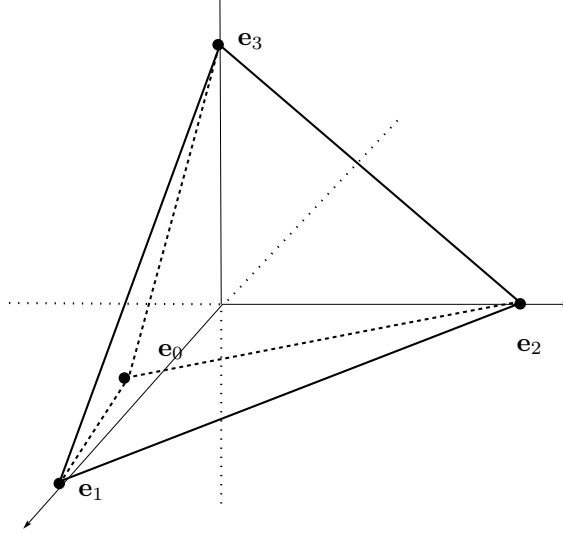


Figure 5: Foliation cones

5.2 The Foliation Cones

Bases of the foliation cones are given in Figure 5 and can be found by doing the four simple disk decompositions using the disks $\{D_1, D_2, D_3\}$, $\{D_0, D_2, D_3\}$, $\{D_1, D_0, D_3\}$, and $\{D_1, D_2, D_0\}$. Thus every lattice point in the four open cones of Figure 5 correspond to depth one foliations. The foliation cones obtained this way are seen to be maximal by the Markov processes argument of [5, §7].

Remark The face Q^+ (respectively Q^-) meets the interior of both $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$ and $\langle \mathbf{e}_0, \mathbf{e}_2, \mathbf{e}_3 \rangle$ (respectively $\langle \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_3 \rangle$ and $\langle \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2 \rangle$). Thus none of the foliation cones can be the union of cones over faces of the Thurston ball.

Remark In this example it is not true that the Thurston ball of DM is the double cone (suspension) of $B/2$. The dimension of $H_2(DM, \partial DM)$ is 4 and $x(\pm \mathbf{R}/2) = 1$ but the two exceptional faces, coned with $\pm \mathbf{R}/2$ do not give faces of the unit ball. The cones over the other faces are faces of the unit ball.

References

- [1] **A Candel** and **L Conlon**, *Foliations I*, Graduate Studies in Mathematics, Vol. 23, Amer. Math. Soc., Providence, 2001.

- [2] **A Candel and L Conlon**, *Foliations II*, Graduate Studies in Mathematics, Vol. 60, Amer. Math. Soc., Providence, 2004.
- [3] **J Cantwell and L Conlon**, *Isotopy of depth one foliations*, Proceedings of the International Symposium and Workshop on Geometric Study of Foliations, Tokyo, World Scientific, November 1993, pp. 153–173.
- [4] **J Cantwell and L Conlon**, *Surgery and foliations of knot complements*, Journal of Knot Theory and its Ramifications 2 (1993), 369–397.
- [5] **J Cantwell and L Conlon**, *Foliation cones*, Geometry and Topology Monographs, Proceedings of the Kirbyfest, vol. 2, 1999, pp. 35–86.
- [6] **J Cantwell and L Conlon**, *Foliation cones revised*, (in preparation).
- [7] **D Gabai**, *Foliations and Genera of Links*, Topology 23 (1984), 381–394.
- [8] **D Gabai**, *Foliations and the topology of 3-manifolds*, J. Diff. Geo. 18 (1983), 445–503.
- [9] **A Hatcher**, *Algebraic Topology*, Cambridge University Press,
- [10] **M Scharlemann**, *Sutured manifolds and generalized Thurston norms*, J. Diff. Geo. 29 (1989), 557–614.
- [11] **W Thurston**, *A norm on the homology of three-manifolds*, Mem. Amer. Math. Soc. 59 (1986), 99–130.